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Self-dual Chern–Simons equations and Nambu–Goto action

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Abstract. Geometrically induced flat $SO(3)$ connections on a string world-sheet are used to construct a self-dual Chern–Simons system for minimal and constant mean curvature surfaces immersed in R^3. Instanton-type solutions are shown to correspond to vortex configurations. Inclusion of a third fundamental form is discussed.

The gauged nonlinear Schrodinger equation in (2+1) dimensions with a non-Abelian Chern–Simons gauge field has been investigated by Grossman [1], Dunne *et al* [2] and Dunne [3], as a generalization of the Abelian Chern–Simons matter-gauge dynamics first proposed by Jackiw and Pi [4]. The gauge group considered in [1–3] is $SU(N)$. The coupling of the matter and gauge fields is done through the Chern–Simons equation

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho} J^{\rho} \tag{1}$$

which dynamically determines the gauge field by a covariantly conserved current J^{μ}. Equation (1) and the gauged nonlinear Schrodinger equation can be obtained from a Lagrangian and can be written as

$$\mathrm{i}\partial_t \psi = \frac{\delta H}{\delta \psi^{\dagger}}$$

where ψ is in the $(SU(N))$ Lie algebra and the Hamiltonian H is

$$H = -\int \mathrm{d}^2 x\, \mathrm{Tr}(D_+ \psi^{\dagger})(D_- \psi)$$

with $D_{\pm} = D_1 \pm \mathrm{i} D_2$; $D_{\mu} = \partial_{\mu} + [A_{\mu},\ ]$. Static solutions are then equivalent to minimizing the energy functional H. The energy-minimizing equation (zero energy)

$$D_{\pm} \psi = 0 \tag{2}$$

and the constraint (1)

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = [\psi^{\dagger}, \psi] \tag{3}$$

where ρ of J^{μ} has been expressed in terms of ψ, are collectively referred to as self-dual Chern–Simons equations. In [1, 3], all finite charge $SU(N)$ solutions have been classified and are shown to yield two-dimensional classical Toda equations.

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The important observation of Grossman [1] is the self-dual Chern–Simons equations (2) and (3) corresponding to the gauge group $SU(N)$ can be obtained as a dimensional reduction of the four-dimensional $SU(N)$ self-dual equation

$$G_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$$

where $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]$, by identifying $W_{1,2} = A_{1,2}$ and $W_3 + \mathrm{i}W_4 = \psi$; $W_3 - \mathrm{i}W_4 = \psi^\dagger$ and with all the W_μ fields depending on x_1 and x_2 only (this is the dimensional reduction) [5]. It will be useful to consider the generalization of the results of [1, 3] to Lie algebras other than $SU(N)$.

It is the purpose of this paper to show that the energy-minimizing equation (2) and the constraint (1) or (3) can be obtained from the Nambu–Goto (NG) action for a string theory, when the string world-sheet (Euclideanized) is regarded as a Riemann surface conformally immersed in R^n ($n \geqslant 3$). The appropriate gauge group is $SO(2) \times SO(n-2)$ dictated by the geometry of the world-sheet, corresponding to the $SO(2)$ symmetry of the tangent frame and the $SO(n-2)$ symmetry of the normal frame. The (NG) action is

$$S_{\mathrm{NG}} = \int \sqrt{g}\,\mathrm{d}^2\xi \tag{4}$$

where $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) is the induced metric on the world-sheet,

$$g_{\alpha\beta} = \partial_\alpha X^\mu(\xi)\partial_\beta X_\mu(\xi) \tag{5}$$

with $X^\mu(\xi)$ as the coordinates of a point on the world-sheet: $\mu = 1$ to n, and ξ_1, ξ_2 are the local coordinates on the world-sheet. Regarded as an immersed surface, $X^\mu(\xi_1, \xi_2)$ are the immersion coordinates. We [6] have proposed a formalism to study the dynamics of the string world-sheet using the generalized Gauss map [7]. The Gauss map is defined as

$$\mathcal{G} : M_0 \to G_{2,n} \simeq SO(n)/(SO(2) \times SO(n-2)) \tag{6}$$

where M_0 is the world-sheet and the Grassmannian $G_{2,n}$ arises due to two tangents and $(n-2)$ normals to the surface. $G_{2,n}$ admits a complex structure and we introduce $z = \xi_1 + \mathrm{i}\xi_2$, $\bar{z} = \xi_1 - \mathrm{i}\xi_2$. It is convenient to regard $G_{2,n}$ as a quadric Q_{n-2} in CP^{n-1} [7]. As $G_{2,n}$ is a set of two-planes in R^n passing through an origin, the (local) tangent two-plane to M_0 becomes an element of $G_{2,n}$ or equivalently a point in Q_{n-2}. Then

$$\partial_z X^\mu(z, \bar{z}) = \psi\Phi^\mu \tag{7}$$

where $\Phi^\mu \in Q_{n-2}$; $\Phi^\mu\Phi_\mu = 0$ and ψ is a complex function of z and $\bar{z}$, determined by the geometrical properties of M_0 [7]. As not every element of $G_{2,n}$ is a tangent plane to M_0, the Gauss map (6) or (7) has to satisfy $(n-2)$ conditions of integrability. For the pupose of this paper, we consider the immersion of the string world-sheet in R^3. In this case, Φ^μ is parametrized as

$$\Phi^\mu = \{1 - f^2, \mathrm{i}(1 + f^2), 2f\} \tag{8}$$

where $f = f(z, \bar{z})$ is a CP^1-field. (8) satisfies $\Phi^\mu\Phi_\mu = 0$. The Gauss map integrability condition in terms of f is

$$\mathrm{Im}\left(\frac{f_{z\bar{z}}}{f_{\bar{z}}} - \frac{2\bar{f} f_z}{1 + |f|^2}\right)_{\bar{z}} = 0. \tag{9}$$

For an immersed surface, besides the induced metric (5) (the first fundamental form), there will be the second fundamental form [8], $H^i_{\alpha\beta}$ ($i = 1$ to $n-2$) and for immersion in R^3, the scalar mean curvature $h = \frac{1}{2}g^{\alpha\beta}H_{\alpha\beta}$ is related to f by

$$(\ell n h)_z = \frac{f_{z\bar{z}}}{f_{\bar{z}}} - \frac{2\bar{f} f_z}{1 + |f|^2} \tag{10}$$

the Kenmotsu equation [9]. The reality of h leads to (9). The real unit normal to the surface is

$$N^\mu = \frac{1}{1+|f|^2}\{f+\bar{f}, -\mathrm{i}(f-\bar{f}), |f|^2-1\}. \tag{11}$$

The *energy integral* of the surface [7] is

$$\mathcal{E} = \int \frac{|f_{\bar{z}}|^2+|f_z|^2}{(1+|f|^2)^2}\frac{\mathrm{i}}{2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \tag{12}$$

and the equation of motion minimizing $\mathcal{E}$ is

$$f_{z\bar{z}} - \frac{2\bar{f} f_z f_{\bar{z}}}{1+|f|^2} = 0. \tag{13}$$

If the Gauss map function f in (7) through (8) satisfies (13), then the surface has constant mean curvature scalar (h) in view of (10).

In the study of string dynamics, using the generalized Gauss map, it follows from (5), (7) and $\Phi^\mu\Phi_\mu = 0$, that the induced metric is destined to be in the conformal gauge. In terms of the complex coordinates, then, $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = |\psi|^2|\Phi|^2 = 2|\psi|^2(1+|f|^2)^2 = \sqrt{g}$. Using the expression for ψ [6], it follows that the NG action can be written as

$$S_{\mathrm{NG}} = \sigma\int \frac{|f_{\bar{z}}|^2}{h^2(1+|f|^2)^2}\frac{\mathrm{i}}{2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \tag{14}$$

where σ is the string tension. Suppose we consider the world-sheet as a Riemann surface of constant mean curvature scalar h, then absorbing the constant h^2 in σ, the NG action (14) becomes equivalent to the *energy integral* of the surface modulo the degree of the map from $S^2 \to CP^1$. We will consider such constant mean curvature surfaces hereafter. Then (13) is the equation of motion of the NG action as well. Usually the minimum of the NG action is taken to correspond to minimal surface ($h = 0$) and in this case the classical action is taken to be $\mathcal{E}$ in (12). Then the CP^1-field f is holomorphic and once again the equation of motion (13) is satisfied by $f = f(z)$. For minimal surfaces, there is no integrability condition on f. Both classes (minimal and constant mean curvature) of world-sheets have been studied earlier [10]. In both cases the equation of motion is (13), the difference being $f = f(z)$ for minimal surfaces and $f = f(\bar{z})$ for the other case.

Earlier [11] the real tangents $\hat{e}_{1,2}$ and the $(n-2)$ normals to the world-sheet have been constructed in terms of the Gauss map, and the antisymmetric $(n\times n)$ matrices A_z and $A_{\bar{z}}$ defined as

$$\partial_z\hat{e}_i = (A_z)_{ij}\hat{e}_j \qquad i, j = 1 \text{ to } n \tag{15}$$

where $\hat{e}_3, \hat{e}_4, \ldots \hat{e}_n$ are the $(n-2)$ normals (a similar definition with z replaced by $\bar{z}$ gives $A_{\bar{z}}$), have been shown to transform as $SO(3, C)$ gauge fields under local $SO(n)$ transformations on $\hat{e}_i$. These non-Abelian gauge fields are geometrically induced and are characteristics of the world-sheet. Furthermore the field strength associated with them is zero:

$$\partial_{\bar{z}}A_z - \partial_z A_{\bar{z}} + [A_{\bar{z}}, A_z] = 0 \tag{16}$$

as can be seen from (15) and we thus have geometrically-induced zero curvature $SO(n, C)$ gauge fields on the world-sheet. For $n = 3$, from (8) and (11) it follows that

$$A_z = \frac{1}{(1+|f|^2)}\begin{bmatrix} 0 & -\mathrm{i}(f\bar{f}_z - \bar{f} f_z) & -(f_z+\bar{f}_z) \\ \mathrm{i}(f\bar{f}_z - \bar{f} f_z) & 0 & \mathrm{i}(f_z - \bar{f}_z) \\ f_z + \bar{f}_z & -\mathrm{i}(f_z - \bar{f}_z) & 0 \end{bmatrix}. \tag{17}$$

$A_{\bar{z}}$ can be obtained by replacing z by $\bar{z}$, $f_z = \frac{\partial f}{\partial z}$ and it can be seen that $A_{\bar{z}} = -(A_z)^\dagger$.

The gauge fields $A_z(A_{\bar{z}})$ are projected onto $SO(2)\times SO(n-2)$ and the coset $G_{2,n}$. Considering the local gauge group associated with $SO(n)$, we have

$$M_0 \ni (z,\bar{z}) \xrightarrow{g} g(z,\bar{z}) \in SO(n) \tag{18}$$

where $g(z,\bar{z})$ can be explicitly written from $\hat{e}_i$ $(i=1,2,\ldots n)$ as an $(n\times n)$ matrix whose ith column is $\hat{e}_i$. Then from (15)

$$A_{z(\bar{z})} = -g^{\dagger}\partial_{z(\bar{z})}g. \tag{19}$$

Under a local gauge transformation generated by $u(z,\bar{z}) \in SO(2)\times SO(n-2)$,

$$\begin{aligned} &g(z,\bar{z}) \rightarrow g(z,\bar{z})u(z,\bar{z}) \\ &A_{z(\bar{z})} \rightarrow u^{\dagger}A_{z(\bar{z})}u + u^{\dagger}\partial_{z(\bar{z})}u \end{aligned} \tag{20}$$

i.e. $A_{z(\bar{z})}$ transform as gauge fields under $SO(2)\times SO(n-2)$ gauge transformation as well.

We now project $A_{z(\bar{z})}$ onto $SO(2)\times SO(n-2)$ and the coset $G_{2,n}$. Denoting the generators of the Lie algebra of $SO(n)$ by $L(\tilde{\sigma})$, those of $SO(2)\times SO(n-2)$ by $L(\bar{\sigma})$ and the remaining by $L(\sigma)$, we have [12]

$$\begin{aligned} a_\alpha(g) &= L(\bar{\sigma})\,\mathrm{Tr}(L(\bar{\sigma})A_\alpha) \\ b_\alpha(g) &= L(\sigma)\,\mathrm{Tr}(L(\sigma)A_\alpha) \end{aligned} \tag{21}$$

as projections onto $SO(2)\times SO(n-2)$ and the coset respectively. In (21) α stands for z or $\bar{z}$. It is easily seen that under the local $SO(2)\times SO(n-2)$ gauge transformation (20)

$$\begin{aligned} &a_\alpha(g) \rightarrow a_\alpha(gu) = u^{\dagger}a_\alpha u + u^{\dagger}\partial_\alpha u \\ &b_\alpha \rightarrow b_\alpha(gu) = u^{\dagger}b_\alpha u \end{aligned} \tag{22}$$

namely, that a_α transforms as an $SO(2)\times SO(n-2)$ gauge field and b_α transforms homogeneously. In this way the geometrically induced $SO(n,C)$ gauge fields $A_{z(\bar{z})}$ on the world-sheet are decomposed into $SO(2)\times SO(n-2)$ gauge fields $a_{z(\bar{z})}$ and a field $b_{z(\bar{z})}$ transforming homogeneously under $SO(2)\times SO(n-2)$ gauge transformation.

The central theme of this paper is to obtain equations (2) and (3) (self-dual Chern–Simons equations) from a_α and b_α defined in (21) and from (15). The zero-curvature equation (16) for $A_{z(\bar{z})}$ when recast in terms of a_α and b_α (21), on account of the Lie group structure of these fields, gives rise to two equations

$$\begin{aligned} &\partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z + [a_z, a_{\bar{z}}] = -[b_z, b_{\bar{z}}] \\ &D_{\bar{z}} b_z = D_z b_{\bar{z}} \end{aligned} \tag{23}$$

where $D_{z(\bar{z})} = \partial_{z(\bar{z})} + [a_{z(\bar{z})},]$. For $n=3$, denoting the anti-Hermitian generators of $SO(3)$ by $T_1, T_2, T_3; [T_1, T_2] = T_3$ (cyclic), we have $a_{z(\bar{z})} = \frac{1}{2}T_3\,\mathrm{Tr}(T_3 A_{z(\bar{z})})$ and $b_{z(\bar{z})} = \frac{1}{2}T_1\,\mathrm{Tr}(T_1 A_{z(\bar{z})}) + \frac{1}{2}T_2\,\mathrm{Tr}(T_2 A_{z(\bar{z})})$. These are explicitly given in terms of the CP^1-field f as

$$\begin{aligned} a_z &= \frac{1}{(1+|f|^2)}\begin{bmatrix} 0 & \mathrm{i}(f\bar{f}_z - \bar{f}f_z) & 0 \\ -\mathrm{i}(f\bar{f}_z - \bar{f}f_z) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ b_z &= \frac{1}{(1+|f|^2)}\begin{bmatrix} 0 & 0 & f_z + \bar{f}_z \\ 0 & 0 & -\mathrm{i}(f_z - \bar{f}_z) \\ -(f_z + \bar{f}_z) & \mathrm{i}(f_z - \bar{f}_z) & 0 \end{bmatrix} \end{aligned} \tag{24}$$

exhibiting the Lie structure mentioned above. The fields $a_{\bar{z}}$ and $b_{\bar{z}}$ are obtained by replacing z in (24) above by $\bar{z}$. Now $D_z b_{\bar{z}}$ can be evaluated using (24) and it is found to be vanishing if the equation of motion (13) is used. In this way, we have

$$\begin{aligned} &\partial_z a_{\bar z} - \partial_{\bar z} a_z + [a_z, a_{\bar z}] = -[b_z, b_{\bar z}] \\ &D_z b_{\bar z} = 0 \\ &D_{\bar z} b_z = 0 \end{aligned} \tag{25}$$

which are identical to (2) and (3) upon identifying ψ with b_z. This procedure can be extended directly to surfaces immersed in R^n ($n \geqslant 4$) with the realization of the set of equations (25). Considering the case with $n = 3$, in analogy with [3] the matter density ρ can be identified with $[b_z, b_{\bar z}] = \rho_3 T_3$.

We now examine solutions to the self-dual equations obtained in the case of a string world-sheet immersed in R^3. We first note that equations (25) are obtained by using the equation of motion (13) and so it is pertinent to concentrate on the possible solutions to (13). Possible solutions to (13) correspond to particular classes of surfaces (world-sheets). There are two cases of interest: (i) in view of (10) it follows that the surface has a constant mean curvature scalar if (13) is satisfied. In the Gauss map description [7], where $f_{\bar z} \neq 0$, such surfaces are described by the holomorphic function, $f = f(\bar z)$. This solves (13) and the integrability condition as well. (ii) Minimal surfaces correspond to $h = 0$ and there is no integrability condition on f [7]. f is taken to be anti-holomorphic $f = f(z)$ which satisfies (13). While constant mean curvature surfaces correspond to the dynamics of closed strings, minimal surfaces correspond to open strings. In both cases we find

$$\partial_{\bar z} \partial_z \ell n \rho_3 = \pm \rho_3 \tag{26}$$

the Liouville equation similar to equation (15) of Dunne [3]. The general solution for ρ_3 is then $\pm\frac{1}{2}\partial_{\bar z}\partial_z \ell n(1 + |f|^2)$ with $f = f(\bar z)$ for case (i) and $f = f(z)$ for case (ii). It is possible to choose explicitly the solution $f(z)$ and $f(\bar z)$ as instantons and anti-instantons. Then

$$f(z) = c\frac{\prod_{i=1}^N (z - a_i)}{\prod_{i=1}^N (z - b_i)} \tag{27}$$

where $\{c, a_i, b_i\}$ are the (complex) instanton parameters, representing the Gauss map of a minimal surface with $2N$ punctures [10]. For anti-instantons,

$$f(\bar z) = \bar c\frac{\prod_{i=1}^N (\bar z - \bar a_i)}{\prod_{i=1}^N (\bar z - \bar b_i)}. \tag{28}$$

The solutions (27) and (28) satisfy the equation of motion (13) and the self-dual system (25). Evaluating $[b_z, b_{\bar z}]$, we find from (25)

$$\partial_z a_{\bar z} - \partial_{\bar z} a_z + [a_z, a_{\bar z}] = -2\frac{(|f_{\bar z}|^2 - |f_z|^2)}{(1 + |f|^2)^2} T_3 \tag{29}$$

the magnetic field lying in the Cartan subalgebra of $SO(3)$. Integrating (29) over the whole surface, we obtain the magnetic flux along T_3 as

$$\phi = \pm 4\pi N \tag{30}$$

for solutions (27) and (28), thereby describing a magnetic vortex with vorticity 4π, similar to the recent observation of Nardeli [13].

The case of the world-sheet having constant mean curvature is of special interest in what follows. In the considerations of immersed surfaces, Eisenhart [8] introduced the *third fundamental form,*

$$\mathcal{H}_{\alpha\beta} = \sum_{i=1}^{n-2} H^i_{\alpha\gamma} H^{i\gamma}_{\beta} \tag{31}$$

where $H^i_{\alpha\beta}$ are the $(n-2)$ second fundamental forms $(N^{i\mu}\partial_\alpha\partial_\beta X^\mu)$ or components of extrinsic curvature, $\mathcal{H}_{\alpha\beta}$ is symmetric in α and β and $\alpha, \beta = z$ or $\bar{z}$. The line element on the immersed surface has been generalized [8] to

$$\mathrm{d}s^2 = (g_{\alpha\beta} + \mathcal{H}_{\alpha\beta})\,\mathrm{d}\xi^\alpha\,\mathrm{d}\xi^\beta \tag{32}$$

so that a notion of 'generalized metric' $\mathcal{G}_{\alpha\beta} = g_{\alpha\beta} + \mathcal{H}_{\alpha\beta}$ is now available. This permits a generalization of the NG action as

$$\int \sqrt{\mathcal{G}}\,\mathrm{d}^2\xi = \int \sqrt{g_{\alpha\beta} + \mathcal{H}_{\alpha\beta}}\,\mathrm{d}^2\xi \tag{33}$$

in the form inspired by Born-Infeld action. Note, however, that $\mathcal{H}_{\alpha\beta}$ is symmetric in α and β. Note also that the above action (33) is Weyl-invariant and contains in its lowest order the NG and extrinsic curvature action of Polyakov [14]. It will be interesting to examine (33) to know (cf (4)) whether its classical equations also yield self-dual Chern–Simons equations.

We consider immersion in R^3. Using the Gauss map, it is straightforward to evaluate (31). The various components of $\mathcal{H}_{\alpha\beta}$ are

$$\begin{aligned}
\mathcal{H}_{zz} &= 4g^{z\bar{z}}\psi f_z(\bar{\psi}\bar{f}_z + \psi f_{\bar{z}})\\
\mathcal{H}_{z\bar{z}} &= 4g^{z\bar{z}}\bar{\psi}(\psi f_z \bar{f}_{\bar{z}} + \bar{\psi}\bar{f}_z\bar{f}_z)\\
\mathcal{H}_{\bar{z}z} &= 4g^{z\bar{z}}\psi(\bar{\psi}\bar{f}_{\bar{z}} f_z + \psi f_{\bar{z}} f_{\bar{z}})\\
\mathcal{H}_{\bar{z}\bar{z}} &= 4g^{z\bar{z}}\bar{\psi}\bar{f}_{\bar{z}}(\psi f_{\bar{z}} + \bar{\psi}\bar{f}_z).
\end{aligned} \tag{34}$$

For constant mean curvature surfaces described by $f = f(\bar{z})$, $\mathcal{H}_{zz} = \mathcal{H}_{\bar{z}\bar{z}} = 0$ and $\mathcal{H}_{z\bar{z}} = 4g^{z\bar{z}}\bar{\psi}^2\bar{f}_z\bar{f}_z$; $\mathcal{H}_{\bar{z}z} = 4g^{z\bar{z}}\psi^2 f_{\bar{z}} f_{\bar{z}}$. From the Gauss map relation, $\psi = -\bar{f}_z/(h(1+|f|^2)^2)$, it follows that

$$\int \sqrt{\mathcal{G}}\,\mathrm{d}^2\xi = \left(2 + \frac{2}{h^2}\right)\int \frac{f_{\bar{z}}\bar{f}_z}{(1+|f|^2)^2}\frac{\mathrm{i}}{2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \tag{35}$$

and this action is equivalent to the NG action (14) expressed in terms of f up to a constant multiplicative factor. So the equation of motion of (33) will be the same as (13) and hence exhibit the self-dual Chern–Simons system (25) as before. In other words, when constant mean curvature surfaces are considered, two actions (4) and (33), have the same equations of motion and exhibit the same self-dual system of equations.

To summarize: utilizing the group structure of immersed surfaces, we have made use of the geometrically induced $SO(3)$ flat connections to construct a self-dual Chern–Simons system of equations, for minimal and constant mean curvature surfaces. The NG action describing the string world-sheet provides the necessary equation of motion to realize the above system. Instanton-type solutions are shown to correspond to vortex configurations. Inclusion of a third fundamental form is shown to have the same feature. This study complements the investigation of [3] for Lie algebras other than $SU(N)$.

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